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X-*s*-SEMIPERMUTABLE SUBGROUPS OF FINITE GROUPS

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Abstract

Let X be a non-empty subset of a group G. A subgroup A of G is said to be X-s-semipermutable in G, if A has a supplement T in G such that for every Sylow subgroup T_p of T there exists an element $x \in X$ such that $AT_p^x = T_p^x A$. In this paper, we use X-s-semipermutable subgroup to study the structures of some finite groups.

1. Introduction

All groups considered in this paper are finite.

A subgroup A of a group G is said to be *permutable* with a subgroup B, if $AB = BA$. A subgroup A is said to be *permutable* or *quasinormal* [2, 17] in G, if A is permutable with all subgroups of G. The permutable subgroup has many interesting properties, for example, Ore [17] proved that every permutable subgroup of a group is subnormal. Itô and Szép

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[14] proved that for every permutable subgroup H of a group G, H / H_G is nilpotent. Kegel [15] and Deskins [1] proved that the subgroup H permute with all Sylow subgroups of G is also subnormal. Recently, Guo et al. [6, 9, 10] introduced the concept of conditionally permutable subgroup (in more generality, X-quasinormal subgroup for some non-empty subset X of G). They say that subgroups H and T of a group G are conditionally permutable (X-permutable) in G, if $HT^x = T^x H$ for some $x \in G$, (respectively, $x \in X$). Some later, they introduced again the concept of X-semipermutable subgroup [8, 16]. For a non-empty subset X of a group G, a subgroup A of G is said to be *X-semipermutable* in G, if A has a supplement T in G such that A is X-permutable with any subgroup of T. By using these concepts, some interesting results have been obtained (see, for example, [5-12, 16]).

As a continuation, we now give the following definition.

Definition 1.1. Let A, B be subgroups of a group G and X be a non-empty subset of G. Then A is said to be *X-s-semipermutable* in G, if A has a supplement T in G such that A is X-permutable with all Sylow subgroups of T.

Throughout this paper, we will use $X_s(A)$ to denote the set of all such supplements T of A in G that A is X-permutable with all Sylow subgroups of T. Obviously, A is X-s-semipermutable in G, if and only if $X_s(A) \neq \varnothing$.

In the present paper, we study the influence of X-s-semipermutable subgroups on the structure of finite groups.

All unexplained notation and terminology are standard. The reader is referred to [3, 18].

2. Preliminaries

In this section, we give some basic properties of X-s-semipermutable subgroups and also cite some known results, which are used in the sequel.

Lemma 2.1. *Let A and X be subgroups of G and* $N \trianglelefteq G$. *Then*:

(1) *If H is a permutable subgroup of G and A is X-s-semipermutable in G, then H A is an X-s-semipermutable subgroup of G.*

(2) *If A is X-s-semipermutable in G, and* $T \in X_s(A)$, *then* AN/N *is* XN/N*-s-semipermutable in* G/N, *and* $TN/N \in (XN/N)_s(AN/N)$.

(3) *If* A/N *is* XN/N*-s-semipermutable in* G/N, *and* $T/N \in (XN/N)_s(A/N)$, *then A is X-s-semipermutable in G and* $T \in X_s(A)$.

(4) *If A is X-s-semipermutable in* G, $A \leq D \trianglelefteq G$, *and* $X \leq D$, *then A is X-s-semipermutable in D.*

(5) *If* $T \in X_s(A)$ *and* $A \leq N_G(X)$, *then* $T^x \in X_s(A)$, *for any* $x \in G$.

(6) *If A is X-s-semipermutable in G and* $X \leq D$, *then A is D-s-semipermutable in G.*

Proof. (1) Let $T \in X_s(A)$. Then $G = AT = HAT$ and for any Sylow subgroup T_p of T, we have $AT_p^x = T_p^x A$, for some $x \in X$. This implies that $HAT_p^x = T_p^x HA$ and also $T \in X_s(HA)$.

(2) Obviously, $G/N = (A/N)(TN/N)$. Let $p \in \pi(TN/N)$ and T_pN/N be a Sylow p-subgroup of TN/N, where T_p be a Sylow p subgroup of T. Then $AT_p^x = T_p^x A$, where $x \in X$. It follows that $(AN/N)(T_pN/N)^{Nx} = (T_pN/N)^{Nx}(AN/N)$. Therefore, $TN/N \in (XN/N)_s(AN/N)$.

(3) Since $T/N \in (XN/N)_s(A/N), G/N = (A/N)(T/N) = AT/N$, and so $G = AT$. Take T_p be a Sylow p-subgroup of T for any $p \in \pi(T)$. Then T_pN/N be a Sylow p-subgroup of T/N and $(A/N)(T_pN/N)^{Nx}$ $= (T_pN/N)^{Nx}(A/N)$, for some $Nx \in XN/N$. This implies that $AT_p^x = T_p^x A$ and $x \in X$. Thus $T \in X_s(A)$.

(4) Let T be a supplement of A in G and $T \in X_s(A)$. Then $D = D \cap AT = A(D \cap T)$. Take a Sylow p-subgroup P of $D \cap T$, for any $p \in \pi(D \cap T)$. Then there exists a Sylow p-subgroup T_p of T such that $P = D \cap T_p$. Since $AT_p^x = T_p^x A$ for some $x \in X$, $AP^x = A(D \cap T_p)^x = D \cap AT_p^x = D \cap T_p^x A = (D \cap T_p)^x A = P^x A$. Thus A is X-s-semipermutable in D.

(5) Since $T \in X_s(A), G = AT = AT^x$, for any $x \in G$ (see the following Lemma 2.2) and so T^x is also a supplement of A in G. It is easy to see that $T^x = T^a$ for some $a \in A$. Let P be a Sylow p-subgroup of T^a, where $p \in \pi(T^a)$. Then $P^{a^{-1}} \in Syl_p(T)$. Thus, there exists an element $d \in X$ such that $A(P^{a^{-1}})^d = (P^{a^{-1}})^d A$. Then $[A(P^{a^{-1}})^d]^a = [(P^{a^{-1}})^d A]^a$ $A^a(d^{-1})^a(P^{a^{-1}})^a d^a = AP^{d^a} = P^{d^a}A$, and $d^a \in X$ since $A \in N_G(X)$. This shows that $T^x \in X(A)$.

(6) This part is evident.

Lemma 2.2 [4, Lemma 3.1]**.** *Let N and L be normal subgroups of a group G. Let P/L be a Sylow p-subgroup of NL/L, and M/L be a maximal subgroup of P/L. If P_p is a Sylow p-subgroup of $P \cap N$, then P_p is a Sylow p-subgroup of N such that $D = M \cap N \cap P_p$ is a maximal subgroup of P_p and $M = LD$.*

Lemma 2.3 [9, Lemma 3.10]**.** *Let G be a group and p, q be distinct prime divisors of $|G|$. Let P be a noncyclic Sylow p-subgroup of G. If every maximal subgroup of P has a q-closed supplement in G, then G is q-closed.*

Lemma 2.4 [9, Lemma 3.13]**.** *Let A and B be subgroups of a group G such that $G \neq AB$ and $AB^x = B^x A$ for each $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.*

Lemma 2.5. *Let G be a group, P be a p-subgroup of G, and Q be a q-subgroup of G, where* p, q *are different primes dividing* $|G|$. *If L is a subnormal subgroup of G and* $PQ = QP$, *then* $PQ \cap L = (P \cap L)(Q \cap L)$.

Proof. It follows directly from [3, Lemma 3.8.2].

3. Main Results

Theorem 3.1. *Let G be a group and* $X = F(G)$. *Then G is supersoluble, if and only if every maximal subgroup of each noncyclic Sylow subgroup of G having no supersoluble supplement in G is X-s-semipermutable in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency part.

Assume that the assertion is false and let G be a counterexample of minimal order. Then:

(1) *The quotient group* G/R *is supersoluble for any minimal normal subgroup R of G.*

Assume that P/R is a noncyclic Sylow p-subgroup of G/R, and P_1/R is a maximal subgroup of P/R. If G_p is a Sylow p-subgroup of P, then G_p is a noncyclic Sylow p-subgroup of G and $G_pR/R = P/R$. By Lemma 2.2, $P_1 \cap G_p$ is a maximal subgroup of G_p and $P_1 = R(P_1 \cap G_p)$. If $P_1 \cap G_p$ has a supersoluble supplement T in G. Then $TR/R \cong T/T \cap R$ is a supersoluble supplement of P_1/R in G/R. If $P_1 \cap G_p$ is X-s-semipermutable in G, then by Lemma 2.1(2), $R(P_1 \cap G_p)/R = P_1/R$ is XR/R-s-semipermutable in G/R. This shows that the hypotheses holds for the quotient group G/R. Thus G/R is supersoluble by the choice of G.

(2) *G has a unique minimal normal subgroup H and* $\Phi(G) = 1$.

It is clear by (1), since the class of all supersoluble groups is a saturated formation.

(3) *G is soluble and so $G = [H]M$, where $H = C_G(H) = O_p(G) = F(G) = X$ is not cyclic and M is a maximal supersoluble subgroup of G.*

Suppose that G is not soluble. Then $X = F(G) = 1$ by (1). Let p be the smallest prime divisor of $|H|$ and H_p be a Sylow p-subgroup of H. It is clear that H_p is not a cyclic group (see [18, II, Theorem 5.5]). Take some Sylow p-subgroup P of G such that $H_p \subseteq P$. Then P is not cyclic. If every maximal subgroup of P possesses a supersoluble supplement in G, then by Lemma 2.3, G is q-closed, where q is the largest prime divisor of $|G|$, which contradicts $F(G) \neq 1$. Therefore, by the hypothesis, there is a maximal subgroup P_1 of P, which is X-s-semipermutable in G. Let $T \in X_s(P_1)$. Then for any Sylow q-subgroup Q of T, where $q \neq p$, we have $P_1Q = QP_1$. Let L be a minimal subnormal subgroup of G. Then L is a simple non-abelian group. Obviously for any $a \in L$, we have $Q^a \in Syl_q(T^a)$. Moreover, Lemma 2.1(5) implies that $T^a \in X_s(P_1)$ and so $P_1Q^a = Q^aP_1$. By Lemma 2.5, $P_1Q^a \cap L = (P_1 \cap L)(Q^a \cap L) = (P_1 \cap L)(Q \cap L)^a$. Obviously, $(P_1 \cap L)(Q \cap L)^a \neq L$ since the left is soluble. It follows from Lemma 2.4 that L is not a simple group, which is a contradiction. Thus G is a soluble group. The rest part is clear.

(4) *H is not a Sylow subgroup of G.*

Assume that H is a Sylow subgroup of G. Clearly, H is not a cyclic subgroup. Let H_1 be a maximal subgroup of H. By the hypothesis, H_1 either possesses a supersoluble supplement T in G or is X-s-semipermutable in G. In the first case, $H = H \cap H_1T = H_1(H \cap T)$ and consequently $H \cap T \neq 1$. Moreover, since H is abelian, obviously $H \cap T \trianglelefteq HT = G$ and so $H \cap T = H$, which implies that $T = G$ is supersoluble, a contraction. Therefore, we may assume that there exists a subgroup $T \in X_s(H_1)$, that is, $G = H_1T$, and for any $q \neq p$ and any

Sylow q-subgroup Q of T, we have $H_1Q^x = Q^xH_1$ for some $x \in X = H$. It follows that $H_1Q = QH_1$ since $H = C_G(H)$. Now, since H_1 is a subnormal Sylow subgroup of H_1Q, $Q \leq N_G(H_1)$. This induces that $H_1 \trianglelefteq G$, which contradicts the minimal normality of H.

(5) *Final contradiction.*

Let q be the largest prime divisor of $|G|$. Suppose that $p = q$. Then since G / H is supersoluble and $H = O_p(G)$, H is a Sylow p-subgroup of G, a contradiction. Hence, $p < q$. Let P be a Sylow p-subgroup (clearly, P is not cyclic) and Q be a Sylow q-subgroup of G. It is easy to see that every maximal subgroup P_1 of P containing H has a supersoluble supplement in G. If every maximal subgroup of P not containing H also has a supersoluble supplement T in G, then by Lemma 2.3, $Q \trianglelefteq G$ and so $Q \leq F(G) = H$, a contradiction. This shows that P has a maximal subgroup P_1 such that $P = HP_1$ and P_1 is X-s-semipermutable in G. Take a subgroup $T \in X_s(P_1)$, then $G = P_1T$ and for any Sylow r-subgroup R of T, where $r \neq p$, we have $P_1R^x = R^xP_1$ for some $x \in X$. Let $D = P_1\langle R^x \mid R \in Syl_r(T),\, r \neq p\rangle$, then $G = HD$ and $H \cap D \trianglelefteq G$. Obviously, $D \cap H \neq H$ and so $H \cap D = 1$. It follows that $H \cap P_1 = 1$, and thereby $|H| = |P : P_1| = p$. This contradiction completes the proof.

Theorem 3.2. *Let G be a group and $X = F(G)$. Then the following are equivalent*:

(1) *Every maximal subgroup of each Sylow subgroup of G is X-s-semipermutable in G.*

(2) $G = [D]M$ *is a supersoluble group, where D and M are nilpotent Hall subgroups of G, and every maximal subgroup of D is normal in G.*

Proof. (1) $\Rightarrow$ (2). Assume that this is not true and let G be a counterexample of minimal order. Then:

(1) *G is supersoluble* (It follows directly from Throrem 3.1) *and clearly G is not nilpotent*.

(2) *Let* $D = G^{\mathfrak{N}}$ *is the nilpotent residual of G. Then D is a nilpotent Hall subgroup of G.*

Obviously, $D \subseteq G'$. Since G is supersoluble, $G' \subseteq F(G)$, and hence $D \subseteq X = F(G)$. We now prove that D is a Hall subgroup of G. We first suppose that G has two minimal normal subgroups H and R such that H is a p-subgroup and R is a q-group with $p \neq q$. Without loss of generality, we may suppose that $H \subseteq D$. Using the same argument as in Theorem 3.1, we know that the hypothesis is true for G / R. Hence, by the choice of G, we have $(G / R)^{\mathfrak{N}} = G^{\mathfrak{N}} R / R = DR / R$ is a Hall subgroup of G / R. Let D_p be a Sylow p-subgroup of D, then $D_p R / R$ is a Sylow p-subgroup of DR / R. Consequently, $D_p R / R$ is a Sylow p-subgroup of G / R. Therefore, D_p be a Sylow p-subgroup of G. Assume that $D_p \neq D$ and let D_r be a Sylow r-subgroup of D, where $r \neq p$. We consider the factor group G / H. Then we conclude, as above, that D_r be a Sylow r-subgroup of G. This implies that D is a Hall subgroup of G. Now, consider the case that all the minimal normal subgroups of G are p-groups. In this case, by virtue of the supersolubility of G, $F(G) = O_p(G)$ is a Sylow p-subgroup of G and so $D \subseteq O_p(G)$. Let H be a minimal normal subgroup of G, if $H \neq D$, then by induction D is a Sylow p-subgroup of G since G / H satisfies the hypothesis.

Now, it remains only to consider the case $H = D$. If $\Phi = \Phi(O_p(G)) \neq 1$, then obviously G / Φ satisfies the hypothesis and so $\Phi D / \Phi = \Phi G^{\mathfrak{N}} / \Phi = (G / \Phi)^{\mathfrak{N}}$ is a Hall subgroup of G / Φ. If $H \subseteq \Phi$, then $G / \Phi \cong (G / G^{\mathfrak{N}}) / (\Phi / G^{\mathfrak{N}})$ is nilpotent, which leads to that G is nilpotent since $\Phi \subseteq \Phi(G)$. The contradiction shows that $H \not\subseteq \Phi$ and so $H\Phi / \Phi$ is a non-identity p-group. Since, $(G / \Phi)^{\mathfrak{N}} = G^{\mathfrak{N}} \Phi / \Phi = H\Phi / \Phi$

is a Hall subgroup of G/Φ, $H\Phi/\Phi$ is a Sylow p-subgroup of G/Φ. It follows that $H\Phi = O_p(G)$ and so $H = D = O_p(G)$ is a Sylow subgroup of G.

Now, we assume that $\Phi(O_p(G)) = 1$, and then $O_p(G) = F(G) = C_G(F(G))$ is an elementary abelian p-group. We show that every proper subgroup T of $O_p(G) = F(G)$ is normal in G. Assume that T is a maximal subgroup of $O_p(G)$. Since $O_p(G)$ is a Sylow p-subgroup of G, by the hypothesis, T is X-s-semipermutable in G. By using the same argument as in proof of Theorem 3.1, we can see that $T \trianglelefteq G$. Assume that T is a non-maximal subgroup of $O_p(G)$. We prove that T is also normal in G. Clearly, it needs only to show that T is the intersection of all maximal subgroups T_i of $O_p(G)$ containing T. Since, $\Phi(O_p(G)) = 1$, $O_p(G)$ is an elementary abelian p-subgroup. Hence, $O_p(G)/T$ is also an elementary abelian subgroup and so $\Phi(O_p(G)/T) = 1$. It follows that T is the intersection of all maximal subgroups of $O_p(G)$ containing T.

By [3, Theorem 1.8.17], $O_p(G) = \langle a \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$, where $\langle a_i \rangle$ is a minimal normal subgroup of G. Let $\langle a \rangle = H$ and $a_1 = aa_2 \cdots a_t$, then clearly $O_p(G) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_t \rangle$. Because G is not nilpotent and $D = G^{\mathfrak{N}} \subseteq O_p(G)$, $O_p(G) \not\subseteq Z(G)$. Hence, there exists an index i such that $a_i \notin Z(G)$. Then G has an element g such that $(|g|, p) = 1$ and $g \notin C_G(a_i)$. Let $y = [[a_i, y_1], \cdots y_n]$, where $y_1 = \cdots = y_n = g$, and $n = c(G/D)$ is the nilpotent class of G/D. Then $y \in H = D$. On the other hand, $y \in \langle a_i \rangle$ and $y \neq 1$ since $g \notin C_G(a_i)$. Hence $\langle a_i \rangle = H$, which is impossible. This completes the proof of (2).

(3) $G = [D]M$, *where D and M are nilpotent Hall subgroups of G.* It follows directly from (2) and the well known Schur-Zasenhaus theorem.

(4) *To completes the proof of* (1) $\Rightarrow$ (2).

Let M be an arbitrary maximal subgroup of D. Then $|D : M| = p$, for some prime p. Let M_p be a Sylow p-subgroup of M. Then $M = M_p M_{p'}$, where $M_{p'}$ is a Hall p'- subgroup of D. Since $M_{p'}$ char $D \trianglelefteq G$, $M_{p'} \trianglelefteq G$. Thus, in order to show that $M \trianglelefteq G$, we only need to prove that M_p is normal in G. It is clear that M_p is a maximal subgroup of the Sylow p-subgroup D_p of D. By the hypothesis, there exists a subgroup $T \in X_s(M_p)$, that is, for any Sylow q-subgroup Q of T, there exists $x \in X$ such that $M_p Q^x = Q^x M_p$. Clearly, $M_p \trianglelefteq D_p$ char $D \trianglelefteq G$. This implies that M_p is a subnormal Hall subgroup of $M_p Q^x$, and consequently $Q^x \in N_G(M_p)$. Moreover, obviously $M_p \trianglelefteq D_p$. Thus $M_p \trianglelefteq G$. This completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Let $p \in \pi(G)$ and P_1 be a maximal subgroup of a Sylow p-subgroup of G. We prove that P_1 is X-s-semipermutable in G. Firstly, it is clear that G is itself a supplement of P_1 in G. Let Q be an arbitrary Sylow q-subgroup of G and $\pi_1 = \pi(D)$, $\pi_2 = \pi(M)$. We now distinguish the following possible cases:

(i) $q \in \pi_1$. In this case, $Q \trianglelefteq G$ and so $P_1 Q = QP_1$.

(ii) $p = q \in \pi_2$. In this case, $P = Q^y$ for some $y \in G$. Consequently, $P_1 Q^y = Q^y P_1$. Let $\Sigma = \{Q_1, \cdots, Q_n\}$ be a Sylow systems of G containing Q and $N = N_G(\Sigma)$. By [13, VI, Theorem 11.10], N covers all central principal factors of G. Since G is supersoluble, $G' \subseteq X = F(G)$. Thus $G = G'N = XN$. Now let $y = xn$, where $x \in X$ and $n \in N$, then we have $P_1 Q^x = Q^x P_1$.

(iii) $p, q \in \pi_2$ and $p \neq q$. Since G is soluble, there exists $y \in G$ such that $PQ^y = Q^y P$. Let $D_1 = PQ^y$. Then $D_1 \leq M^g$ for some $g \in G$. Since M is nilpotent, D_1 is nilpotent. Hence $Q^y \trianglelefteq D_1$ and thereby, $P_1 Q^y =$

$Q^y P_1$. Using the same arguments in the proof of (ii), one can show that $P_1 Q^x = Q^x P_1$ for some $x \in X$.

(iv) $p \in \pi_1$, $q \in \pi_2$. In this case, $P \trianglelefteq G$. Let $D_{p'}$ be a Hall p'-subgroup of D. Then $D_{p'} \trianglelefteq G$ since D is nilpotent. Obviously, $M_1 = P_1 D_{p'}$ is a maximal subgroup of D. Consequently, $M_1 \trianglelefteq G$. Therefore, $P \cap M_1 = P \cap P_1 D'_p = P_1(P \cap D'_p) = P_1 \trianglelefteq G$. This implies that $P_1 Q = Q P_1$.

Thus, the proof is completed.

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